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Combinatorial identities and quantum state densities of supersymmetric sigma models on N-folds

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Abstract. There is a remarkable connection between the number of quantum states of conformal theories and the sequence of dimensions of Lie algebras. In this paper, we explore this connection by computing the asymptotic expansion of the elliptic genus and the microscopic entropy of black holes associated with (supersymmetric) sigma models. The new features of these results are the appearance of correct prefactors in the state density expansion and in the coefficient of the logarithmic correction to the entropy.

1 Introduction

The combinatoric identities with which we shall be concerned play an important role in a number of physical models. In particular, such identities can be associated with the elliptic genus partition function of supersymmetric sigma models on the N-folds and play a special role in string and black hole dynamics. Before entering into the specific problem, we would like to introduce the reader to some formal aspects found in the mathematical literature [1,2], giving two examples in string theory which we shall use later on.

Let us first discuss CY_3-folds: we consider type IIA string theory compactified on Calabi–Yau three-folds CY_3. Recently, black holes have been studied in the $N=2$ supergravity corresponding to type IIA strings. This theory can also be viewed as M-theory on $CY_3 \times S^1$ and extremal black holes are microscopically represented by fivebranes wrapping on $P \times S^1$, where P is a four-cycle, $P \subset CY_3$. The microscopic entropy of the fivebrane has been determined from a two-dimensional (0,4) sigma model, whose target space includes the fivebrane moduli space [3]. We are interested in the CY_3 geometry. The charge forms are in one to one correspondence with the elements of $H^q(CY_3, \mathcal{L} \otimes \Omega^p)$, where $\mathcal{L}$ is the line bundle for which $c_1(\mathcal{L}) = [F_{CY}]$; $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. $H^q(CY_3, \mathcal{L} \otimes \Omega^p)$ vanishes for $q>0$ and large $c_1(\mathcal{L})$. One can compute $\dim H^0(CY_3, \mathcal{L} \otimes \Omega^p)$ using the Riemann–Roch formula, and the result is (see also [4])

$$
\begin{aligned}
h_0 &= \dim H^0(CY_3, \mathcal{L}) \\
&= \int \left[\frac{F_{CY}^3}{6} + \frac{c_2 \wedge F_{CY}}{12} \right] = \mathcal{D} + \frac{1}{12} c_2 \cdot P, \\
h_1 &= \dim H^0(CY_3, \mathcal{L} \otimes \Omega^1) \\
&= \int \left[\frac{F_{CY}^3}{2} - \frac{3c_2 \wedge F_{CY}}{4} + \frac{c_3}{2} \right] = 3\mathcal{D} - \frac{3}{4} c_2 \cdot P - \frac{\chi}{2}, \\
h_2 &= \dim H^0(CY_3, \mathcal{L} \otimes \Omega^2) \\
&= \int \left[\frac{F_{CY}^3}{2} - \frac{3c_2 \wedge F_{CY}}{4} - \frac{c_3}{2} \right] = 3\mathcal{D} - \frac{3}{4} c_2 \cdot P + \frac{\chi}{2}, \\
h_3 &= \dim H^0(CY_3, \mathcal{L} \otimes \Omega^3) = h_0. \qquad (1.1)
\end{aligned}
$$

Here c_k are the kth Chern classes, χ is the Euler characteristic and P is a four-cycle of a manifold (for more notation and details see for example [3,4]).

Our goal is to compute a partition function $Z(q) \simeq \mathrm{Tr}\,[\mathcal{O} q^N]$ in a black hole geometry, where the trace is calculated over the multibrane Hilbert space and $\mathcal{O}$ is an appropriate operator insertion. We have to count the multiparticle primaries choosing a basis of states. Actually the counting of configurations is in one to one correspondence to the counting of states for conformal field theory with $\sum_{(\text{even } j)} h_j$ bosons and $\sum_{(\text{odd } j)} h_j$ fermions and total momentum N. The number of states would correspond to the coefficient of $D(n)$ in the expansion of the generating function

$$
Z = \mathrm{Tr} q^N = \prod_n \frac{(1+q^n)^{\sum_{(\text{odd } j)} h_j}}{(1-q^n)^{\sum_{(\text{even } j)} h_j}} = \sum_n D(n) q^n. \quad (1.2)
$$

Now, let us have a look at the supersymmetric sigma model on N-folds. That is, our next example is a sigma model on the N-fold symmetric product $S^N X$ of a Kähler

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manifold X, which is the $S^N X = X^N/S_N$ orbifold space. S_N is the symmetric group of N elements. The Hilbert space of an orbifold field theory can be decomposed into twisted sectors $\mathcal{H}_\gamma$, that are labelled by the conjugacy classes $\{\gamma\}$ of the orbifold group S_N [5,6]. For a given twisted sector one can keep the states invariant under the centralizer subgroup Γ_γ related to the element γ. Let $\mathcal{H}_\gamma^{\Gamma_\gamma}$ be an invariant subspace associated with Γ_γ; the total orbifold Hilbert space takes the form $\mathcal{H}(S^N X) = \oplus_{\{\gamma\}} \mathcal{H}_\gamma^{\Gamma_\gamma}$. One can compute the conjugacy classes $\{\gamma\}$ by using a set of partitions $\{N_n\}$ of N, namely $\sum_n nN_n = N$, where N_n is the multiplicity of the cyclic permutation (n) of n elements in the decomposition of γ: $\{\gamma\} = \sum_{j=1}(j)^{N_j}$. For this conjugacy class the centralizer subgroup of a permutation γ is $\Gamma_\gamma = S_{N_1} \otimes_{j=2} (S_{N_j} \rtimes Z_j^{N_j})$ [6], where each subfactor S_{N_n} and Z_n permutes the N_n cycles (n) and acts within one cycle (n) correspondingly. Following the lines of [6] we may decompose each twisted sector $\mathcal{H}_\gamma^{\Gamma_\gamma}$ into a product over the subfactors (n) of N_n-fold symmetric tensor products, $\mathcal{H}_\gamma^{\Gamma_\gamma} = \otimes_{n>0} S^{N_n} \mathcal{H}_{(n)}^{Z_n}$, where $S^N \mathcal{H} \equiv (\otimes^N \mathcal{H})^{S_N}$.

It has been shown that the partition function for (sub) Hilbert space of a supersymmetric sigma model coincides with the elliptic genus [7]. If $\chi(\mathcal{H}_{(n)}^{Z_n}; q, y)$ admits the extension $\chi(\mathcal{H}; q, y) = \sum_{m\geq 0,\ell} G(nm,\ell) q^m y^\ell$, the following result holds [6,8,9]:

$$\prod_{m\geq 0,\ell} \left(1 - pq^m y^\ell\right)^{-G(nm,\ell)} = \sum_{N\geq 0} p^N \chi(S^N \mathcal{H}_{(n)}^{\mathbb{Z}_n}; q, y),$$

$$W(p; q, y) = \prod_{n>0,m\geq 0,\ell} \left(1 - p^n q^m y^\ell\right)^{-G(nm,\ell)}$$

$$= \sum_{N\geq 0} p^N \chi(S^N X; q, y), \tag{1.3}$$

$p = \mathbf{e}[\rho]$, $q = \mathbf{e}[\tau]$, $y = \mathbf{e}[z]$, and $\mathbf{e}[x] \equiv \exp[2\pi i x]$. Here ρ and τ determine the complexified Kähler form and complex structure modulos of $\mathbb{T}^2$ respectively, and z parametrizes the $U(1)$ bundle on $\mathbb{T}^2$. The Narain duality group $SO(3, 2, \mathbb{Z})$ is isomorphic to the Siegel modular group $Sp(4, \mathbb{Z})$ and it is convenient to combine the parameters ρ, τ and a Wilson line modules z into a 2×2 matrix belonging to the Siegel upper half-plane of genus two,

$$\Xi = \begin{pmatrix} \rho & z \\ z & \tau \end{pmatrix},$$

with $\mathrm{Im}\,\rho > 0$, $\mathrm{Im}\,\tau > 0$, $\det \mathrm{Im}\,\Xi > 0$. The group $Sp(4, \mathbb{Z}) \cong SO(3, 2, \mathbb{Z})$ acts on the Ξ matrix by fractional linear transformations $\Xi \to (A\Xi + B)(C\Xi + D)^{-1}$. Note that for a Calabi–Yau space the χ-genus is a weak Jacobi form of zero weight and index $d/2$ [10]. For $q = 0$ the elliptic genus reduces to a weighted sum over the Hodge numbers, namely $\chi(X; 0, y) = \sum_{j,k} (-1)^{j+k} y^{j-\frac{d}{2}} h^{j,k}(X)$. For the trivial line bundle the symmetric product $\prod_{n>0} (1 - p^n)^{-\chi(X)}$ (see Sect. 3 for details) can be associated with the simple partition function of a second quantized string theory.

This paper is organized as follows. In Sect. 2 we discuss the homological method of the relationship between Lie algebras and combinatorial identities following the lines of [1,11]. The asymptotic expansion of the elliptic genus and the microscopic entropy of a black hole associated with a supersymmetric sigma model are given in Sect. 3. The problem of microscopically computing the entropy of black holes has already been solved. In the present paper the new features are the appearance of correct prefactors in the state density expansion and in the coefficient of the logarithmic correction to the entropy. The fact that the computation of the number of states for conformal theories, and therefore the compution of the entropy of black holes, would be connected to the sequence of dimensions of Lie algebras is quite intriguing, and we finish this paper with a discussion of this point.

2 Combinatorial identities and graded Lie algebras

One of the universal methods of obtaining combinatorial identities of the types of (1.2) and (1.3) is the Euler–Poincaré formula associated with a complex consisting of finite-dimensional linear spaces. In this section we briefly discuss the homological aspects of identities following the lines of [1,11]. Our remarks here are designed to provide the readers with a brief introduction to these identities and to indicate how it could be derived from results of (graded) Lie algebras. The relationship between combinatorial identities and Lie algebras was discovered by Macdonald [12] (on the whole all combinatorial identities are related to Lie algebras). In this section we shall apply Euler–Poincaré formula to chain complexes of Lie algebras.

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $C_q(\mathfrak{g})$ be the space of q-dimensional chain of $\mathfrak{g}$. The Euler–Poincaré formula gives

$$\sum_q (-1)^q c_q^{(m)} = \sum_q (-1)^q h_q^{(m)}, \quad m \in \mathbb{N},$$

$$c_q^{(m)} = \dim C_q^{(m)}(\mathfrak{g}), \tag{2.1}$$

$$h_q^{(m)} = \dim H_q^{(m)}(\mathfrak{g}),$$

where $H_q(\mathfrak{g})$ is the homology of the complex $\{C_q(\mathfrak{g})\}$.

Introducing the x variable we can rewrite this sequence of identities as a formal power series:

$$\sum_{q,m} (-1)^q c_q^{(m)} x^m = \sum_{q,m} (-1)^q h_q^{(m)} x^m$$

$$= \prod_j (1 - x^j)^{\dim \mathfrak{g}_{(j)}}. \tag{2.2}$$

Therefore, in order to get the identity in its final form the homology $H_q^{(m)}(\mathfrak{g})$ has to be computed.

Let us suppose that the Lie algebra $\mathfrak{g}$ possesses a polygrading $\mathfrak{g} = \oplus \mathfrak{g}_{(m_1,\ldots,m_k)}$. The following result holds, due to Fuks [1], Theorem 3.2.3.

Theorem 1 *Let*

$$\mathfrak{g} = \oplus_{\substack{m_1 \geq 0, \ldots, m_k \geq 0 \\ m_1 + \ldots + m_k > 0}} \mathfrak{g}_{(m_1, \ldots, m_k)}$$

be the (poly)graded Lie algebra satisfying $\dim \mathfrak{g}_{(m_1,\ldots,m_k)} < \infty$. *If* $\dim H_q^{(m_1,\ldots,m_k)} = h_q^{(m_1,\ldots,m_k)}$, *the formal power series in* $x_1, \ldots, x_k$ *satisfies the following identity*

$$\prod_{j_1,\ldots,j_k} \left(1 - x_1^{j_1} \ldots x_k^{j_k}\right)^{\dim \mathfrak{g}_{(m_1,\ldots,m_k)}} = \sum_{q, m_1, \ldots, m_k} (-1)^q h_q^{m_1,\ldots,m_k} x_1^{m_1} \ldots x_k^{m_k}. \qquad (2.3)$$

We can apply Theorem 1 to compute the Lie algebras homology which has been carried out in the papers [13,14]. Assume that a Hermitian or Euclidean metric can be chosen in every space $\mathfrak{g}_{(\lambda)}$. As a consequence, all the spaces $C^q_{(\lambda)}(\mathfrak{g})$ acquire a metric and one can identify $C^q_{(\lambda)}(\mathfrak{g})$ with $[C^q_{(\lambda)}(\mathfrak{g})]'$, i.e. with $C_q^{(\lambda)}(\mathfrak{g})$. The homology of the Lie algebras can be entirely computed by using the Laplace operator $\mathfrak{L}^q_{(\lambda)}$. Every element of the space $H^q_{(\lambda)}(\mathfrak{g})$ can be represented by an unique harmonic cocycle from $C^q_{(\lambda)}(\mathfrak{g})$ (see, for example, [1]). It means that there is a natural isomorphism:

$$\mathrm{Ker}\, \mathfrak{L}^q_{(\lambda)} = H^q_{(\lambda)}(\mathfrak{g}). \qquad (2.4)$$

As an example, let us briefly consider an application of Theorem 1 to graded Lie subalgebras $N^+(\mathfrak{g}^A)$ of Kac–Moody algebras. The algebra $N^+(\mathfrak{g}^A)$ is constructed from the Cartan matrix A. The matrix $A = ||a_{ij}||^n_{i,j=1}$ is square integer, $a_{ij} = 2$, $i, j = 1, 2, \ldots, n$, $a_{ij} \leq 0$ for $i \neq j$, and for which there exist positive numbers $b_1, \ldots, b_n$, such that the matrix $bA = ||b_i a_{ij}||$ is symmetric. The Kac–Moody algebra $\mathfrak{g}^A$ with Cartan matrix A is the complex Lie algebra with generators $e_1, \ldots, e_n$. We can construct $\mathbb{Z}^n$-grading in the Kac–Moody algebra $\mathfrak{g}^A$, choosing the appropriate form of $\{\deg e_j\}^n_{j=1}$. The following theorem occurs, also due to Fuks [1].

Theorem 2 *Equation* (2.3) *applied to graded Kac–Moody algebras gives*

$$\prod_{\substack{k_1 \geq 0, \ldots, k_n \geq 0 \\ k_1 + \ldots + k_n > 0}} (1 - x_1^{k_1} \ldots x_n^{k_n})^{\dim \mathfrak{g}^A_{(k_1,\ldots,k_n)}} = \sum_{Q(j_1,\ldots,j_n)=0} L(j_1, \ldots, j_n) x_1^{j_1} \ldots x_n^{j_n}, \qquad (2.5)$$

where $L(j_1, \ldots, j_n)$ *are certain coefficients.*

The identities which correspond to the $N^+(\mathfrak{g}^A)$ algebra with Cartan matrices of rank $n-1$ with negative eigenvalues are usually called Macdonald identities. Combinatorial identities related to Kac–Moody algebras with other Cartan matrices are also of interest as we will now see.

3 Asymptotics of generating functions

One can apply the Theorem 1 to the $\mathbb{Z}^n$-grading of the $\mathfrak{sl}(n, \mathbb{C})$ subalgebras of Lie algebras together with Theorem 2. It gives the Macdonald identities series containing the Gauss–Jacobi identity (see for details [1]):

$$\prod_{m=1}^{\infty} (1 - x_1^m x_2^m)(1 - x_1^m x_2^{m-1})(1 - x_1^{m-1} x_2^m) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(x_1^{\frac{k(k+1)}{2}} x_2^{\frac{k(k-1)}{2}} + x_1^{\frac{k(k-1)}{2}} x_2^{\frac{k(k+1)}{2}} \right).$$

We can make use of trivial transformations of the Gauss–Jacobi identity. It reduces to the series

$$\prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{k=1}^{\infty} (-1)^{k-1} (2k-1) x^{\frac{k(k-1)}{2}}, \qquad (3.1)$$

which is the cube of the "Euler function" $\prod_{n=1}^{\infty}(1 - x^n)$. Interesting combinatorial identities may be obtained by applying Theorem 1 to graded Lie algebras, but only for some chosen values of the power k can the function $\prod_{n=1}^{\infty}(1-x^n)^k$ be presented by a power of Euler function. Some arguments on "distinguished powers" (which is in correspondence to the sequence of dimensions of Lie algebras) the reader can find in [15] (Sects. 3 and 4).

Since the coefficient $D(n)$ in the expansion of the generating function in its final form is not always known, we shall simplify the calculations and apply its asymptotic limit. The multi-component version of the classical generating functions has the form

$$\mathfrak{G}_{\pm}(z) = \prod_{\mathbf{n} \in \mathbb{Z}^p / \{\mathbf{0}\}} \left[1 \pm \exp\left(-z \omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g})\right)\right]^{\pm\sigma}, \qquad (3.2)$$

where $\mathrm{Re}\, z > 0$, $\sigma > 0$, $\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g})$ is given by $\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g}) = \left(\sum_\ell a_\ell (n_\ell + \mathrm{g}_\ell)^2\right)^{1/2}$, g_ℓ, and a_ℓ are some real numbers (for arbitrary spectral forms $\omega^2_{\mathbf{n}}$ see, for example, [16]). The total number of quantum states can be described by the functions $D_{\pm}(n)$ defined by $\mathcal{K}_{\pm}(t) = \sum_{n=0}^{\infty} D_{\pm}(n) t^n \equiv \mathfrak{G}_{\pm}(-\log t)$, where $t < 1$, and n is a total quantum number. The p-dimensional Epstein zeta function $Z_p \left|{}^{\mathbf{g}}_{\mathbf{f}}\right| (z, \varphi)$ associated with the quadratic form $\varphi[\mathbf{a}(\mathbf{n} + \mathbf{g})] = (\omega_{\mathbf{n}}(\mathbf{a}, \mathbf{g}))^2$ for $\mathrm{Re}\, z > p$ is given by the formula

$$Z_p \begin{vmatrix} \mathrm{g}_1 \ldots \mathrm{g}_p \\ f_1 \ldots f_p \end{vmatrix} (z, \varphi) = \sum_{\mathbf{n} \in \mathbb{Z}^p} {}' \left(\varphi[\mathbf{a}(\mathbf{n} + \mathbf{g})]\right)^{-\frac{z}{2}} \mathrm{e}^{2\pi \mathrm{i} (\mathbf{n}, \mathbf{f})}, \qquad (3.3)$$

where $(\mathbf{n}, \mathbf{f}) = \sum_{i=1}^p n_i f_i$, f_i are real numbers and the prime on $\sum{}'$ means that one should omit the term $\mathbf{n} = -\mathbf{g}$ if all the g_i are integers. For $\mathrm{Re}\, z < p$, $Z_p \left|{}^{\mathbf{g}}_{\mathbf{f}}\right| (z, \varphi)$ is understood to be the analytic continuation of the right hand side of (3.3). Note that $Z_p \left|{}^{\mathbf{g}}_{\mathbf{f}}\right| (z, \varphi)$ is an entire function in the complex z-plane except for the case when all the f_i are integers. In this case $Z_p \left|{}^{\mathbf{g}}_{\mathbf{f}}\right| (z, \varphi)$ has a simple pole at $z = p$ with residue $A(p) = 2\pi^{p/2}[(\det \mathbf{a})^{1/2}\Gamma(p/2)]^{-1}$, which does not depend on the winding numbers g_ℓ.

By means of the asymptotic expansion of $\mathfrak{G}_\pm(z)$ for small z, one arrives at a complete asymptotic limit of $D_\pm(n)$ [17–19]:

$$D_\pm(n)_{n\to\infty} = \mathcal{C}_\pm(p) n^{\frac{2\sigma Z_p|^{\mathbf{g}}_{\mathbf{0}}|(0,\varphi)-p-2}{2(1+p)}} \times \exp\left\{\frac{1+p}{p}[\sigma A(p)\Gamma(1+p)\zeta_\pm(1+p)]^{\frac{1}{1+p}} n^{\frac{p}{1+p}}\right\} \times[1+\mathcal{O}(n^{-k_\pm})], \quad (3.4)$$

$$\mathcal{C}_\pm(p) = [\sigma A(p)\Gamma(1+p)\zeta_\pm(1+p)]^{\frac{1-2\sigma Z_p|^{\mathbf{g}}_{\mathbf{0}}|(0,\varphi)}{2p+2}} \times \frac{\exp\left[\sigma(\mathrm{d}/\mathrm{d}z)Z_p\,|^{\mathbf{g}}_{\mathbf{0}}|\,(z,\varphi)|_{(z=0)}\right]}{[2\pi(1+p)]^{1/2}}, \quad (3.5)$$

where $\zeta_-(z) \equiv \zeta_{\mathrm{R}}(z)$ is the Riemann zeta function,

$$\zeta_+(z) = (1-2^{1-z})\zeta_{\mathrm{R}}(z),$$
$$k_\pm = p/(1+p)\min\left(\mathcal{C}_\pm(p)/p - \delta/4, 1/2-\delta\right),$$

and $0 < \delta < 2/3$.

3.1 Asymptotics of the elliptic genus

If $y = \mathbf{e}[z] = 1$ then the elliptic genus degenerates to the Euler number or Witten index [20]. For the symmetric product this gives the following identity:

$$W(p) = \sum_{N\geq 0} p^N \chi(S^N X) = \prod_{n>0}(1-p^n)^{-\chi(X)}. \quad (3.6)$$

Thus this character is almost a modular form of weight $-\chi(X)/2$. Equation (3.6) is similar to the denominator formula of a (generalized) Kac–Moody algebra [21]. A denominator formula can be written as follows:

$$\sum_{\sigma\in\mathcal{W}}(\mathrm{sgn}(\sigma))\,\mathrm{e}^{\sigma(v)} = \mathrm{e}^v \prod_{r>0}(1-\mathrm{e}^r)^{\mathrm{mult}(r)}, \quad (3.7)$$

where v is the Weyl vector, the sum on the left hand side is over all elements of the Weyl group $\mathcal{W}$, the product on the right hand side runs over all positive roots (one has the usual notation of root spaces, positive roots, simple roots and Weyl group, associated with the Kac–Moody algebra) and each term is weighted by the root multiplicity $\mathrm{mult}(r)$. For the $su(2)$ level, for example, an affine Lie algebra (3.7) is just the Jacobi triple product identity. For generalized Kac–Moody algebras there is the following denominator formula:

$$\sum_{\sigma\in\mathcal{W}}(\mathrm{sgn}(\sigma))\,\sigma\left(\mathrm{e}^v\sum_r \varepsilon(r)\mathrm{e}^r\right) = \mathrm{e}^v\prod_{r>0}(1-\mathrm{e}^r)^{\mathrm{mult}(r)}, \quad (3.8)$$

where the correction factor on the left hand side involves $\varepsilon(r)$ which is $(-1)^n$ if r is the sum of n distinct pairwise orthogonal imaginary roots and zero otherwise.

The logarithm of the partition function $W(p;q,y)$ is the one-loop free energy $F(p;q,y)$ for a string on $\mathbb{T}^2\times X$:

$$\begin{aligned} F(p;q,y) &= \log W(p;q,y) = -\sum_{n>0,m,\ell} G(nm,\ell)\log\left(1-p^n q^m y^\ell\right)\\ &= \sum_{n>0,m,\ell,k>0}\frac{1}{k}G(nm,\ell)p^{kn}q^{km}y^{k\ell}\\ &= \sum_{N>0}p^N\sum_{kn=N}\frac{1}{k}\sum_{m,\ell}G(nm,\ell)q^{km}y^{k\ell}. \end{aligned} \quad (3.9)$$

The free energy can be written as a sum of Hecke operators T_N acting on the elliptic genus of X [6,21,22]: $F(p;q,y) = \sum_{N>0}p^N T_N\chi(X;q,y)$.

The goal now is to calculate an asymptotic expansion of the elliptic genus $\chi(S^NX;q,y)$. The degeneracies for the sigma model are given by the Laurent inversion formula: $\chi(S^NX;q,y) = (2\pi\mathrm{i})^{-1}\oint W(p,q,y)p^{-N-1}\mathrm{d}p$, where the contour integral is taken on a small circle around the origin. Let the Dirichlet series

$$D(s;\tau,z) = \sum_{(n,m,\ell)>0}\sum_{k=1}^{\infty}\frac{\mathbf{e}[\tau mk + z\ell k]G(nm,\ell)}{n^s k^{s+1}} \quad (3.10)$$

converge for $0<\mathrm{Re}\,s<\alpha$. We assume that the series (3.10) can be analytically continued in the region $\mathrm{Re}\,s \geq -C_0$ ($0<C_0<1$) where it is analytic except for a pole of order one at $s=0$ and $s=\alpha$, with residue $\mathrm{Res}[D(0;\tau,z)]$ and $\mathrm{Res}[D(\alpha;\tau,z)]$ respectively. Besides, let $D(s;\tau,z) = \mathcal{O}(|\,\mathrm{Im}\,s|^{C_1})$ uniformly in $\mathrm{Re}\,s\geq -C_0$ as $|\,\mathrm{Im}\,s|\to\infty$, where C_1 is a fixed positive real number. The Mellin–Barnes representation of the function $F(t;\tau,z)$ has the form

$$\hat{M}[F](t;\tau,z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,s=1+\alpha} t^{-s}\Gamma(s)D(s;\tau,z)\mathrm{d}s. \quad (3.11)$$

The integrand in (3.11) has a first order pole at $s=\alpha$ and a second order pole at $s=0$. Shifting the vertical contour from $\mathrm{Re}\,s = 1+\alpha$ to $\mathrm{Re}\,s=-C_0$ (this procedure is permissible) and making use of the residue theorem one obtains

$$\begin{aligned} F(t;\tau,z) &= t^{-\alpha}\Gamma(\alpha)\,\mathrm{Res}[D(\alpha;\tau,z)] + \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}[sD(s;\tau,z)]\\ &\quad -(\gamma+\log t)\,\mathrm{Res}[D(0;\tau,z)]\\ &\quad +\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{Re}\,s=-C_0}t^{-s}\Gamma(s)D(s;\tau,z)\mathrm{d}s, \end{aligned} \quad (3.12)$$

where $t\equiv 2\pi(\mathrm{Im}\,\rho - \mathrm{i}\,\mathrm{Re}\,\rho)$. The absolute value of the integral in (3.12) can be estimated to behave as $\mathcal{O}\left((2\pi\,\mathrm{Im}\,\rho)^{C_0}\right)$.

In the half-plane $\mathrm{Re}\,t>0$ there exists an asymptotic expansion for $W(t;\tau,z)$ uniformly in $|\,\mathrm{Re}\,\rho|$ for $|\,\mathrm{Im}\,\rho|\to 0$, $|\arg(2\pi\mathrm{i}\rho)|\leq\pi/4$, $|\,\mathrm{Re}\,\rho|\leq 1/2$ and given by

$$\log W(t;\tau,z) = \mathrm{Res}[D(\alpha;\tau,z)]\Gamma(\alpha)t^{-\alpha} - \mathrm{Res}[D(0;\tau,z)]\log t - \gamma\mathrm{Res}[D(0;\tau,z)] + \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}[sD(s;\tau,z)] + \mathcal{O}\left(|2\pi\,\mathrm{Im}\,\tau|^{C_0}\right). \quad (3.13)$$

Repeating the above, we obtain the result (3.4) and (3.5) with the obvious modifications:

$$\{n;p\} \Longrightarrow \{N;\alpha\}, \quad D_\pm(n) \Longrightarrow \chi(S^N X;\tau,z),$$

and

$$\sigma Z_p \left|{\mathbf{g} \atop \mathbf{0}}\right| (0,\varphi) \Longrightarrow \mathrm{Res}[D(0;\tau,z)],$$
$$\sigma A(p)\zeta_\pm(1+p)\Gamma(1+p) \Longrightarrow \mathrm{Res}[D(\alpha;\tau,z)]\Gamma(1+\alpha),$$
$$\sigma \lim_{z\to 0}\frac{\mathrm{d}}{\mathrm{d}z} Z_p \left|{\mathbf{g} \atop \mathbf{0}}\right| (z,\varphi) \Longrightarrow \lim_{s\to 0}\frac{\mathrm{d}}{\mathrm{d}s}[sD(0;\tau,z)] - \gamma\mathrm{Res}[D(0;\tau,z)], \quad (3.14)$$

where γ is the Euler constant. These results have an universal character for all elliptic genera associated to Calabi–Yau spaces.

Let us note the following. We go into some facts related to orbifoldized elliptic genus of $N = 2$ superconformal field theory. The contribution of the untwisted sector to the orbifoldized elliptic genus is the function $\chi(X;\tau,z) \equiv \Phi_{00}(\tau,z)$, whereas

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \Phi_{00}(\tau,z)\mathbf{e}\left[\frac{rcz^2}{c\tau+d}\right], \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad (3.15)$$

$r = d/2$. The contribution of the twisted μ-sector projected by ν is [8]

$$\Phi_{\mu\nu}(\tau,z) = \Phi_{00}(\tau, z+\mu\tau+\nu)\mathbf{e}\left[d(\mu\nu+\mu^2\tau+2\mu z)/2\right], \quad \mu,\nu\in\mathbb{Z}.$$

For some suitable integers P and ℓ the orbifoldized elliptic genus can be defined by

$$\phi(\tau,z)_{\mathrm{orb}} \stackrel{\mathrm{def}}{=} \frac{1}{\ell}\sum_{\mu,\nu=0}^{\ell-1}(-1)^{P(\mu+\nu+\mu\nu)}\Phi_{\mu\nu}(\tau,z). \quad (3.16)$$

Using the transformation properties of the function $\Phi_{\mu\nu}(\tau,z)$ one can obtain the asymptotic expansion for the orbifoldized elliptic genus. In fact we can introduce a procedure, starting with the expansion of the elliptic genus of the untwisted sector, to compute the asymptotics of the elliptic genus of the twisted sector.

3.2 The microscopic entropy

Results of the previous sections can be used to calculate the ground state degeneracy of systems with quantum numbers of certain states of extreme black holes. We can compute the asymptotics of the functions $\mathfrak{G}_\pm(z)$, $\chi(S^N X;\tau,z)$ associated with a gas of species of massless quanta. In the context of superstring dynamics, for example, the asymptotic state density gives a precise computation of the entropy of a black hole. The black hole entropy $S(N)$ becomes

$$S(N) = \log\chi(S^N X;\tau,z) \simeq S_0 + \mathcal{A}(\alpha)\log(S_0) + (\mathrm{Const.}),$$
$$\mathcal{A}(\alpha) = (2\alpha)^{-1}\{2\,\mathrm{Res}[D(0;\tau,z)] - 2 - \alpha\}. \quad (3.17)$$

The leading term in (3.17) has the form

$$S_0 = B(\alpha)N^{\alpha/(1+\alpha)}, \quad (3.18)$$
$$B(\alpha) = \frac{1+\alpha}{\alpha}\left\{\mathrm{Res}[D(\alpha;\tau,z)]\Gamma(1+\alpha)\right\}^{1/(1+\alpha)}.$$

$\mathcal{A}(\alpha)$ is the coefficient of the logarithmic correction to the entropy; it depends on the complex dimension d of a Kähler manifold X.

In conclusion, we note that the asymptotic state density at level n ($n \gg 1$) for fundamental p-branes compactified on the manifold with topology $\mathbb{T}^p \times \mathbb{R}^{d-p}$ has been calculated within the semiclassical quantization scheme in [17, 19]. In string theory, in the case of zero modes, the embedding spacetime dependence can be eliminated [23], and the coefficient of the logarithmic correction $\mathcal{A}(p)$ becomes $-3/2$, which agrees with the results obtained in the spin network formalism.

To summarize, our results can be used in the context of the brane method to calculate the ground state degeneracy of systems with quantum numbers of certain BPS extreme black holes, for example, the BPS black hole in toroidally compactified type II string theory. One can construct a brane configuration such that the corresponding supergravity solutions describe five-dimensional black holes. Black holes in these theories can carry both an electric charge Q_F and an axion charge Q_H. The brane picture gives the entropy in terms of the partition function $W(t)$ for a gas of Q_FQ_H species of massless quanta: $W(t) = \prod_{\mathbf{n}\in\mathbb{Z}^p/\{\mathbf{0}\}}\left[1-\exp\left(-t\omega_{\mathbf{n}}(\mathbf{a},\mathbf{g})\right)\right]^{-\sigma}$, where $\sigma = (\dim X - p - 1)$, $t = y + 2\pi i x$, $\mathrm{Re}\,t > 0$. For unitary conformal theories of fixed central charge c the entropy becomes $S(n) = \log\chi(n) \simeq S_0 + \mathcal{A}\log(S_0)$, where $S_0 = 2\pi\sqrt{cn/6}$, $\mathcal{A} = -(c+3)/2$. Following [24, 25], we can put $c = 3Q_F^2 + 6, n = Q_H$, and get the growth of the elliptic genus (or the degeneracy of BPS solitons) for $n = Q_H \gg 1$. However, this result is incorrect when the black hole becomes massive enough and its Schwarzschild radius exceeds any microscopic scale such as the compactification radii [26, 27]. Such models, stemming from string theory, would therefore be incompatible; in view of the present result, this might be presented as a useful constraint for the underlying microscopic field theory.

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